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Constitutive equation for a class of isotropic, perfectly elastic solids using a new measure of finite strain and corresponding stress

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Abstract. The definition of a measure of strain, referred to as the bi-configuration strain tensor, centres on the difference between the left Cauchy-Green deformation tensor and its inverse. A new measure of stress, coined the bi-configuration stress tensor, has been defined. This measure of stress refers the traction in the current configuration jointly to the referential and spatial configurations, that is, to an effective element of area identified as an element of bi-configuration area. The stress and strain tensors are assumed to be constitutively related by a finite strain form of a generalised Hooke's law. The predictions obtained from the proposed constitutive equation are compared with the observed mechanical behaviour of various test materials. Comparison with experiment centres on biaxial stress measurements in various simple modes of deformation identified by way of a generalised stress-strain relation. The predictions from the proposed constitutive theory are in good accord with the results of experiment.

Keywords: bi-configuration stress, constitutive equation, finite strain, perfectly elastic, solid material

1. Introduction

A nonlinear constitutive equation for a class of isotropic perfectly elastic solids will have engineering relevance only if it can be shown to uniquely predict the properties characteristic of the mechanical response of typical materials. A distinction must be made between general predictions, which any theoretically admissible constitutive equation must yield, and specific predictions, that is, quantitative predictions that distinguish different constitutive equations and can be subjected to the test of experiment. Any theoretically admissible constitutive equation must predict the Poynting effect [1,2] for a certain class of nonlinear materials. The Poynting effect is thus an example of a general prediction. Simple equibiaxial extension is effectively another general prediction; any theoretically admissible constitutive equation for isotropic, perfectly elastic materials would be expected to predict stress equality for this simple mode of deformation. Examples of constitutive equations for a class of isotropic perfectly elastic solids that give these two general predictions, but do not yield a specific quantitative prediction for finite deformation are the second Piola-Kirchhoff stress – Green-St.Venant strain equation (see, for example [3, pp. 130–132]) and the Cauchy stress – Cauchy-Green deformation equation (see, for example [3, pp. 115–118]).

The present discussion is concerned with a constitutive stress-finite-strain equation that yields quantitative predictions at finite strain, as well as with the comparison of these predictions with experiment. The constitutive equation associates a new measure of stress to a measure of finite strain. Discussion of the properties of the proposed constitutive equation centres on the use of two parameters based on Lode's [4] parameter. These are the Lode stress parameter, which yields predictions that distinguish between different measures of stress, and the Lode strain parameter, which yields predictions that identify the new measure of strain as the only measure of finite strain compatible with the new measure of stress. When expressed in deviatoric form, the constitutive equation establishes equality between the Lode parameters, referred to as the Lode relation. Three simple modes of deformation are characterised by particular values of the Lode parameters: pure shear, uniaxial extension, and equi-biaxial extension. For two of these simple modes of deformation, use of the Lode parameters identifies two quantitative predictions which can be subjected to test using biaxial stress measurements. The constitutive equation predicts the Poynting [1,2] effect.

The discussion is restricted to isotropic perfectly elastic solids, and does not account for thermodynamic restrictions. The proposed constitutive equation is purely mechanical. Furthermore, no attempt is made to take into account sub-continuum, that is the micromechanics of the material. It is, however, this latter aspect of the mechanical properties of materials that has motivated the attempt to develop a constitutive equation whose material coefficients may be tractable to analysis within the general field of the micromechanics of polymeric solids. For this to be possible, the material coefficients must be fundamental physical properties of the material. This is a central consideration of the present discussion.

2. Bi-configuration strain tensor

Using standard notation and conventions, (see, for example [5]), referential Cartesian coordinates denoted $X^\alpha(\alpha = 1, 2, 3)$ and spatial Cartesian coordinates denoted $x^i(i = 1, 2, 3)$ are set up in space by adjoining to the respective origins O and o the similar orthonormal curvilinear bases, $E = (\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3)$ and $e = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$.

The definition of any measure of finite strain must centre on the fundamental kinematic tensors underlying the local analysis of deformation and motion. These fundamental kinematic tensors are the deformation gradient **F** and its inverse defined by

$$\mathbf{F} = \text{Grad}\, \boldsymbol{x}, \quad \mathbf{F}^{-1} = \text{grad}\, \boldsymbol{X}, \tag{2.1}$$

and having, for example, the mixed component form

$$\begin{aligned} &\mathbf{F} = F^p{}_\mu \boldsymbol{e}_p \otimes \boldsymbol{E}^\mu, \quad F^i{}_\alpha = x^i{}_{,\alpha} \\ &\mathbf{F}^{-1} = (F^{-1})^\mu{}_p \boldsymbol{E}_\mu \otimes \boldsymbol{e}^p, \quad (F^{-1})^\alpha{}_i = X^\alpha{}_{,i}\,, \end{aligned} \tag{2.2}$$

where $_{,r}$ $(r = \alpha, i)$ denotes covariant differentiation. Since **F** is invertible, the polar decomposition theorem can be applied to write

$$\mathbf{F} = \mathbf{RU}, \qquad \mathbf{F} = \mathbf{VR}, \tag{2.3}$$

where **R** is the rotation tensor. The positive-definite, symmetric right and left stretch tensors **U** and **V** have the spectral representation

$$\mathbf{U} = \sum_{r=1}^{3} u_r \boldsymbol{p}_r \otimes \boldsymbol{p}_r, \quad \mathbf{V} = \sum_{r=1}^{3} v_r \boldsymbol{q}_r \otimes \boldsymbol{q}_r, \quad \mathbf{V} = \mathbf{RUR}^{\mathrm{T}}, \tag{2.4}$$

where the u_i, v_i are the eigenvalues of **U** and **V**: the orthonormal triplets $\boldsymbol{p}_i, \boldsymbol{q}_i$ define the principal axes of **U** at $\boldsymbol{X}$ and **V** at $\boldsymbol{x}$, respectively. Substituting in the third equation of (2.4) for the left and right stretch tensors yields

$$\mathbf{V} = \sum_{r=1}^{3} v_r \boldsymbol{q}_r \otimes \boldsymbol{q}_r = \sum_{r=1}^{3} u_r \{(\mathbf{R}\boldsymbol{p}_\mathbf{r}) \otimes \boldsymbol{p}_r\}(\boldsymbol{p}_r \otimes q_r) = \sum_{r=1}^{3} u_r \boldsymbol{q}_r \otimes \boldsymbol{q}_r, \tag{2.5}$$

which establishes that the left and right stretch tensors have a common set of eigenvalues referred to as the principal stretch ratios

$$\lambda_i = u_i = v_i \qquad (i = 1, 2, 3). \tag{2.6}$$

Entering the expressions for the right and left stretch tensor given in the first and second equations in (2.4) into the first and second equations in (2.3), respectively, gives for the deformation gradient tensor the expression

$$\mathbf{F} = \sum_{r=1}^{3} \lambda_r \boldsymbol{q}_r \otimes \boldsymbol{p}_r, \quad \lambda_i = \boldsymbol{q}_i \cdot (\mathbf{F}\boldsymbol{p}_i) \quad (i = 1, 2, 3), \quad J = \det \mathbf{F} = \lambda_1\lambda_2\lambda_3 > 0. \tag{2.7}$$

In deriving (2.7), use has been made of (2.6) to replace the common eigenvalues of the stretch tensors by the principal stretch ratios.

The left and right Cauchy-Green deformation tensors **B** and **C** are related to the stretch tensors by the expressions

$$\mathbf{B} = \mathbf{F}\mathbf{F}^\mathrm{T} = \mathbf{V}^2, \quad \mathbf{C} = \mathbf{F}^\mathrm{T}\mathbf{F} = \mathbf{U}^2, \quad \mathbf{B} = \mathbf{R}\mathbf{C}\mathbf{R}^\mathrm{T}, \tag{2.8}$$

where the superscript T denotes the matrix transpose. Using (2.4), (2.6) and (2.8), it follows that **B** and **C** have the spectral representations

$$\mathbf{B} = \sum_{r=1}^{3} \lambda_r^2 \boldsymbol{q}_r \otimes \boldsymbol{q}_r, \quad \mathbf{C} = \sum_{r=1}^{3} \lambda_r^2 \boldsymbol{p}_r \otimes \boldsymbol{p}_r. \tag{2.9}$$

The squares of the principal stretches are the common eigenvalues of **B** and **C**.

For isotropic, perfectly elastic materials, the reference configuration $\mathrm{B_r}$ of the material body $\mathcal{B}$ is the undistorted state to which the material returns when relieved of all stress. In the undistorted state, that is, in the stress free natural state of $\mathcal{B}$, any measure of finite strain must reduce to the zero tensor.

Consider the symmetric second-order tensors

$$\begin{aligned} \mathbf{E} &= \tfrac{1}{4}(\mathbf{B} - \mathbf{B}^{-1}) = \tfrac{1}{4}\left[\mathbf{F}\mathbf{F}^\mathrm{T} - (\mathbf{F}^{-1})^\mathrm{T}\mathbf{F}^{-1}\right], \\ \mathbf{J} &= \tfrac{1}{4}(\mathbf{C} - \mathbf{C}^{-1}) = \tfrac{1}{4}\left[\mathbf{F}^\mathrm{T}\mathbf{F} - \mathbf{F}^{-1}(\mathbf{F}^{-1})^\mathrm{T}\right], \quad \mathbf{E} = \mathbf{R}\mathbf{J}\mathbf{R}^\mathrm{T}, \end{aligned} \tag{2.10}$$

that are rational functions of both **F** and its inverse. For a rigid-body motion of $\mathcal{B}$, the Cauchy-Green deformation tensors reduce to the identity tensor and $\mathbf{E} = \mathbf{O}, \mathbf{J} = \mathbf{O}$. Entering the spectral representations of the left and right Cauchy-Green deformation tensors given in (2.9) into (2.10), we obtain the spectral representation

$$\mathbf{E} = \sum_{r=1}^{3} \varepsilon_r \boldsymbol{q}_r \otimes \boldsymbol{q}_r, \qquad \mathbf{J} = \sum_{r=1}^{3} \varepsilon_r \boldsymbol{p}_r \otimes \boldsymbol{p}_r, \tag{2.11}$$

where

$$\varepsilon_i = \frac{(\lambda_i^4 - 1)}{4\lambda_i^2}(i = 1, 2, 3) \tag{2.12}$$

are the common eigenvalues of **E** and **J**.

The multiplicative decompositions of the strain tensors in terms of the right and left stretch tensors **U** and **V** illustrate the physical significance of **E** and **J**. From (2.8) and (2.10),

$$\mathbf{E} = \mathbf{\Pi P}, \qquad \mathbf{J} = \mathbf{\Theta N}, \tag{2.13}$$

where

$$\mathbf{\Pi} = \tfrac{1}{2}(\mathbf{V} + \mathbf{V}^{-1}) \quad \mathbf{P} = \tfrac{1}{2}(\mathbf{V} - \mathbf{V}^{-1}), \quad \mathbf{\Theta} = \tfrac{1}{2}(\mathbf{U} + \mathbf{U}^{-1}) \quad \mathbf{N} = \tfrac{1}{2}(\mathbf{U} - \mathbf{U}^{-1}). \tag{2.14}$$

In the undistorted state, $\mathbf{P} = \mathbf{O}$ and $\mathbf{N} = \mathbf{O}$. Because calculation of the components of $\mathbf{U}, \mathbf{V}$ and **R** from **F** may in special cases involve irrational functions, the finite strain tensors **P** and **N** are not suitable for the formulation of constitutive equations. The positive-definite, symmetric tensors $\mathbf{\Pi}$ and $\mathbf{\Theta}$ render the tensors **E** and **J** in a form that can be used for this purpose. Hence, although **P** and **N** are the fundamental measures of strain, the term 'finite strain' will be reserved for **E** and **J**. The common eigenvalues ε_i of **E** and **J** are referred to as the principal finite strains. Equations (2.4) and (2.6) can be used to rearrange (2.14) into

$$\kappa_i = \boldsymbol{q}_i \cdot (\mathbf{\Pi}\boldsymbol{q}_i) = \boldsymbol{p}_i \cdot (\mathbf{\Theta}\boldsymbol{p}_i) = \frac{\lambda_i^2 + 1}{2\lambda_i}, \xi_i = \boldsymbol{q}_i \cdot (\mathbf{P}\boldsymbol{q}_i) = \boldsymbol{p}_i \cdot (\mathbf{N}\boldsymbol{p}_i) = \frac{\lambda_i^2 - 1}{2\lambda_i}, \tag{2.15}$$

where the principal strains ξ_i are the common eigenvalues of **P** and **N**; κ_i are the common eigenvalues of $\mathbf{\Pi}$ and $\mathbf{\Theta}$.

Consider a material curve $\mathrm{C_r}$ in the reference configuration $\mathrm{B_r}$ of a material body $\mathscr{B}$, and a material curve $\mathrm{C_t}$ in the current configuration $\mathrm{B_t}$ of $\mathscr{B}$. A material line element in the reference configuration $\mathrm{B_r}$ of $\mathscr{B}$ has length $\mathrm{d}S_{(i)}$ and the direction of a unit vector $\boldsymbol{L}_i$ tangent to $\mathrm{C_r}$ at $\boldsymbol{X}$. This segment is carried into the elementary arc length $\mathrm{d}s_{(i)}$ in the current configuration $\mathrm{B_t}$ of $\mathscr{B}$, and is in the direction of a unit vector $\boldsymbol{l}_i$ tangent to $\mathrm{C_t}$ at $\boldsymbol{x}$. Let the elementary arc lengths $\mathrm{d}S_{(i)}$ and $\mathrm{d}s_{(i)}$ be represented by the vectors $\mathrm{d}\boldsymbol{X}_{(i)}$ and $\mathrm{d}\boldsymbol{x}_{(i)} (i = 1,2,3)$, respectively. These vectors are related through the deformation gradient tensor evaluated at $\boldsymbol{X}$ in the reference configuration $\mathrm{B_r}$ of $\mathscr{B}$ and at $\boldsymbol{x}$ in the current configuration $\mathrm{B_t}$ of $\mathscr{B}$ so that

$$\mathrm{d}\boldsymbol{x}_{(i)} = \mathbf{F}\mathrm{d}\boldsymbol{X}_{(i)} \qquad (i = 1, 2, 3), \tag{2.16}$$

in accordance with the first equation in (2.1). Substituting in (2.16) for $\mathrm{d}\boldsymbol{X}_{(i)} = \boldsymbol{L}_i \mathrm{d}S_{(i)}$ and $\mathrm{d}\boldsymbol{x}_{(i)} = \boldsymbol{l}_i \mathrm{d}s_{(i)}$, and since $\boldsymbol{L}_i$ and $\boldsymbol{l}_i$ each have unit norm, it follows with reference to the second equation in (2.7), that

$$\frac{\mathrm{d}s_{(i)}}{\mathrm{d}S_{(i)}} = \boldsymbol{l}_i \cdot (\mathbf{F}\boldsymbol{L}_i) = \lambda_i \quad (i = 1, 2, 3). \tag{2.17}$$

From (2.17)

$$\lambda_q \lambda_r = \frac{\mathrm{d}s_{(q)}}{\mathrm{d}S_{(q)}} \frac{\mathrm{d}s_{(r)}}{\mathrm{d}S_{(r)}} = \frac{\mathrm{d}a_{(p)}}{\mathrm{d}A_{(p)}} = \frac{J}{\lambda_p} \quad (p, q, r = 1,2,3, p \neq q \neq r), \tag{2.18}$$

where use has been made of the third equation in (2.7) in the form $J = \lambda_p \lambda_q \lambda_r$, and where

$$\mathrm{d}A_{(p)} = \mathrm{d}S_{(q)}\mathrm{d}S_{(r)}, \quad \mathrm{d}a_{(p)} = \mathrm{d}s_{(q)}\mathrm{d}s_{(r)}, \tag{2.19}$$

are elements of area in the referential and spatial configurations of $\mathscr{B}$, respectively. Equations (2.17) and (2.18) give for the principal stretch ratios the expressions

$$\lambda_i = \frac{\mathrm{d}s_{(i)}}{\mathrm{d}S_{(i)}} = J\frac{\mathrm{d}A_{(i)}}{\mathrm{d}a_{(i)}} \quad (i = 1, 2, 3). \tag{2.20}$$

The expression for the principal finite strains ε_i given in (2.12) can be recast into the form

$$\varepsilon_i = \kappa_i \frac{\left(\mathrm{d}s_{(i)}\right)^2 - \left(\mathrm{d}S_{(i)}\right)^2}{2\,\mathrm{d}\sigma_{(i)}} = \kappa_i \frac{\left(J\,\mathrm{d}A_{(i)} + \mathrm{d}a_{(i)}\right)}{2J\,\mathrm{d}\psi_{(i)}} \left[\frac{\Delta\mathcal{A}_{(i)}}{\mathrm{d}\psi_{(i)}}\right] (i = 1{,}2{,}3), \tag{2.21}$$

where use has been made of (2.15) and (2.20), and

$$\mathrm{d}\sigma_{(i)} = \mathrm{d}s_{(i)}\mathrm{d}S_{(i)}, \quad \mathrm{d}\psi_{(i)} = \sqrt{\mathrm{d}A_{(i)}\mathrm{d}a_{(i)}}, \quad \Delta\mathcal{A}_{(i)} = J\,\mathrm{d}A_{(i)} - \mathrm{d}a_{(i)} \quad (i = 1{,}2{,}3). \tag{2.22}$$

The definition of $\mathrm{d}\sigma_{(i)}$ given in the first equation in (2.22) does not admit of a direct physical interpretation. However, $\mathrm{d}\sigma_{(i)}$ and quantities like it will be referred to, solely in the context of physical dimensions, as an element of bi-configuration area, thus emphasizing that $\mathrm{d}\sigma_{(i)}$ is defined jointly on the referential and spatial configurations of $\mathscr{B}$.

Equation (2.21) identifies the physically significant quantities of the finite strain $\mathbf{E}$. These are the change in area $\Delta\mathcal{A}_{(i)}$, which is fundamental to the concept of finite strain, and the element of area $\mathrm{d}\psi_{(i)}$ which, since it is defined jointly on the referential and spatial configurations of $\mathscr{B}$, will be referred to as an element of bi-configuration area. With regard to the definition of the corresponding stress tensor, $\Delta\mathcal{A}_{(i)}$ is the current value of the change in area. Any strain tensor for which the current value of the change in area acts across a surface defined jointly on the referential and spatial configurations of $\mathscr{B}$ will be referred to as a bi-configuration strain tensor.

In the context of the third equation in (2.10), and equations (2.21) and (2.22), $\mathbf{E}$ and $\mathbf{J}$ will be referred to as the left and right bi-configuration strain tensors, respectively.

When the displacements and displacement gradient components are small compared to unity, the same reference axes can be used for $\boldsymbol{x}$ and $\boldsymbol{X}$ to give the displacement vector $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X} = \boldsymbol{u}(\boldsymbol{X})$. This approximation allows the introduction of the tensor of displacement gradients, $\boldsymbol{H} = \boldsymbol{F} - \boldsymbol{I}$, which can be rearranged and entered into the first equation of (2.8) and the inverse of $\mathbf{B}$, to give, to a first order in $\mathbf{H}$, the small strain approximations

$$\mathbf{B} \approx \mathbf{I} + 2\tilde{\mathbf{e}} = \tilde{\mathbf{B}}, \quad \mathbf{B}^{-1} \approx \mathbf{I} - 2\tilde{\mathbf{e}} = \tilde{\mathbf{B}}^{-1}, \quad \tilde{\mathbf{e}} = \tfrac{1}{2}(\mathbf{F} + \mathbf{F}^{\mathrm{T}}) - \mathbf{I}, \tag{2.23}$$

where $\tilde{\mathbf{e}}$ is the classical measure of infinitesimal strain, that is Cauchy's strain measure. Entering the small strain approximations for $\mathbf{B}$ and $\mathbf{B}^{-1}$ given in (2.23) into the first equation in (2.10) gives for the left bi-configuration strain tensor the small strain approximation

$$\mathbf{E} \approx \tfrac{1}{4}(\tilde{\mathbf{B}} - \tilde{\mathbf{B}}^{-1}) = \tilde{\mathbf{e}}. \tag{2.24}$$

Similarly, the small-strain approximation of the right bi-configuration strain tensor is the Cauchy strain tensor.

3. Bi-configuration stress tensor

The symmetric, second-order Cauchy stress tensor $\mathbf{T}$ defined on the spatial configurations of $\mathscr{B}$, has the spectral representation

$$\mathbf{T} = \sum_{r=1}^{3} t_r \boldsymbol{q}_r \otimes \boldsymbol{q}_r, \tag{3.1}$$

where the t_i are the eigenvalues of $\mathbf{T}$. The principal axes of $\mathbf{T}$ at $\boldsymbol{x}$ are defined by the orthonormal triplet $\boldsymbol{q}_i$. Let $\boldsymbol{f}$ be the traction in the current configuration $\mathrm{B_t}$ of $\mathscr{B}$. For the Cauchy stress tensor,

$$\mathrm{d}\boldsymbol{f} = \mathbf{T}^{\mathrm{T}}\boldsymbol{n}\mathrm{d}a. \tag{3.2}$$

Equation (3.2) establishes that the traction is acting across a material surface element with area $\mathrm{d}a$ and outward unit normal vector $\boldsymbol{n}$ in the current configuration $\mathrm{B_t}$ of $\mathscr{B}$. This is in contrast to the left bi-configuration strain tensor $\mathbf{E}$ which is seen from (2.21) to be defined in such a way that the current value of the change in area $\Delta\mathcal{A}_{(i)}$ acts across an element of bi-configuration area $\mathrm{d}\psi_{(i)}$ which is defined jointly on the referential and spatial configurations of $\mathscr{B}$.

From (2.20), the stretch ratios λ_q and λ_r can be expressed in the form

$$\lambda_q = \frac{\mathrm{d}s_{(q)}}{\mathrm{d}S_{(q)}}\frac{\mathrm{d}s_{(r)}}{\mathrm{d}s_{(r)}} = \frac{\mathrm{d}a_{(p)}}{\mathrm{d}\psi_{(p)}}, \qquad \lambda_r = \frac{\mathrm{d}s_{(r)}}{\mathrm{d}S_{(r)}}\frac{\mathrm{d}s_{(q)}}{\mathrm{d}s_{(q)}} = \frac{\mathrm{d}a_{(p)}}{\mathrm{d}\Lambda_{(p)}}, \tag{3.3}$$

where $\mathrm{d}a_{(p)}$ is defined in the second equation in (2.19), and

$$\mathrm{d}\psi_{(p)} = \mathrm{d}S_{(q)}\mathrm{d}s_{(r)}, \quad \mathrm{d}\Lambda_{(p)} = \mathrm{d}s_{(q)}\mathrm{d}S_{(r)}, \tag{3.4}$$

are two elements of bi-configuration area.

A new measure of stress can be defined using (3.3). In this context, (3.3) can be used in two ways to give the expressions

$$\pi_1 = \frac{t_1\mathrm{d}a_{(1)}}{\mathrm{d}\Lambda_{(1)}} = \lambda_3 t_1, \quad \pi_2 = \frac{t_2\mathrm{d}a_{(2)}}{\mathrm{d}\Lambda_{(2)}} = \Lambda_1 t_2, \quad \pi_3 = \frac{t_3\mathrm{d}a_{(3)}}{\mathrm{d}\Lambda_{(3)}} = \lambda_2 t_3, \tag{3.5}$$

and

$$\zeta_1 = \frac{t_1\mathrm{d}a_{(1)}}{\mathrm{d}\psi_{(1)}} = \lambda_2 t_1, \quad \zeta_2 = \frac{t_2\mathrm{d}a_{(2)}}{\mathrm{d}\psi_{(2)}} = \lambda_3 t_2, \quad \zeta_3 = \frac{t_3\mathrm{d}a_{(3)}}{\mathrm{d}\psi_{(3)}} = \lambda_1 t_3. \tag{3.6}$$

However, there is no criterion for selecting preferentially π_i or ζ_i. This is because π_i , ζ_i do not admit a direct physical interpretation. This ambiguity with respect to the physical significance of π_i and ζ_i can be resolved by forming a set of composite eigenvalues $s_i = s_i(\pi_i, \zeta_i)$ $(i = 1,2,3)$. A linear combination of π_i and ζ_i to form the composite eigenvalues s_i would centre on the use of the right and left stretch tensors $\mathbf{U}$ and $\mathbf{V}$. However, while the polar decomposition theorem in the form of (2.3) is central to the proof of general theorems, calculation of the components of $\mathbf{U}$,$\mathbf{V}$ and $\mathbf{R}$ from $\mathbf{F}$ in special cases may involve irrational functions. The π_i and ζ_i can be combined in terms of the squares of the principal stretches, as

$$s_i = \sqrt{\tfrac{1}{2}(\pi_i^2 + \zeta_i^2)} = a_i^{1/2} t_i, \tag{3.7}$$

where

$$a_i = \tfrac{1}{2}(\mathrm{I}_{\mathbf{B}} - \lambda_i^2)\ (u = 1, 2, 3), \qquad \mathrm{I}_{\mathbf{B}} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2. \tag{3.8}$$

Since both $\mathbf{B}$ and $\mathbf{T}$ are symmetric second-order tensors, s_i defined in (3.7) can be taken to be the eigenvalues of the symmetric second-order stress tensor

$$\mathbf{S} = \mathbf{\Phi}\mathbf{T}. \tag{3.9}$$

where

$$\mathbf{\Phi} = \mathbf{A}^{1/2}, \quad \mathbf{A} = \tfrac{1}{2}(\mathrm{I}_{\mathbf{B}}\mathbf{I} - \mathbf{B}). \tag{3.10}$$

With the use of the first equation in (2.9), the positive definite symmetric kinematic tensor $\mathbf{\Phi}$ has the spectral representation

$$\mathbf{\Phi} = \sum_{r=1}^{3} \phi_r \boldsymbol{q}_r \otimes \boldsymbol{q}_r. \tag{3.11}$$

The coefficients

$$\phi_i = \sqrt{\tfrac{1}{2}(\mathrm{I}_{\mathbf{B}} - \lambda_i^2)} = a_i^{1/2} (i = 1, 2, 3) \tag{3.12}$$

are the eigenvalues of $\mathbf{\Phi}$. The expression for $\mathbf{S}$ given in (3.9) is the counterpart of the expression for $\mathbf{E}$ given in the first equation of (2.13).

Entering the representations of $\mathbf{T}$ and $\mathbf{\Phi}$ from (3.1) and (3.11) into (3.9) gives for $\mathbf{S}$ the spectral representation

$$\mathbf{S} = \sum_{r=1}^{3} s_r \boldsymbol{q}_r \otimes \boldsymbol{q}_r, \tag{3.13}$$

where

$$s_i = \phi_i t_i = \sqrt{\frac{(\mathrm{I}_{\mathbf{B}}\lambda_i - \lambda_i^3)}{2J}} \left[\frac{t_i \mathrm{d}a_{(i)}}{\mathrm{d}\psi_{(i)}}\right] (i = 1, 2, 3) \tag{3.14}$$

are the eigenvalues of $\mathbf{S}$. In deriving (3.14) use has been made of (2.20) and (2.22). The expression for s_i given in (3.14) is the counterpart of the expression for the ε_i given in (2.21).

Equations (2.10) - (2.12) imply that, in the context of formulating constitutive equations,

$$\mathbf{S} = \mathbf{R}\mathbf{\Sigma}\mathbf{R}^{\mathrm{T}}. \tag{3.15}$$

This expression for $\mathbf{S}$ is the counterpart of the expression for $\mathbf{E}$ given in the third equation of (2.10). The stress tensor $\mathbf{\Sigma}$ is defined on the reference configuration $\mathrm{B_r}$ of $\mathscr{B}$. Equations (3.9), (3.10) and (3.15) yield

$$\mathbf{\Sigma} = J^{-1}\tilde{\mathbf{\Phi}}\tilde{\mathbf{T}}, \quad \tilde{\mathbf{\Phi}} = \sqrt{\tfrac{1}{2}(\mathrm{I}_{\mathbf{C}}\mathbf{C}^2 - \mathbf{C}^3)}, \quad \tilde{\mathbf{T}} = J\,\mathbf{F}^{-1}\mathbf{T}(\mathbf{F}^{-1})^{\mathrm{T}}, \tag{3.16}$$

where $\tilde{\mathbf{T}}$ is the second Piola-Kirchhoff stress tensor, (see, for example [3, pp. 71–73]). The stress tensors $\mathbf{S}$ and $\mathbf{\Sigma}$ have in common the set of eigenvalues s_i.

In the context of (3.15), $\mathbf{S}$ and $\mathbf{\Sigma}$ will be referred to as the left and right bi-configuration stress tensors, respectively.

For sufficiently small strains, $\mathbf{B} \approx \mathbf{I}$, and hence, from (3.10), $\mathbf{\Phi} \approx \mathbf{I}$, which can be entered into (3.9) to give

$$\lim_{\mathbf{E}\to\tilde{\mathbf{e}}} \mathbf{S} = \mathbf{T}. \tag{3.17}$$

This establishes that, for sufficiently small strains, the left bi-configuration stress tensor approximates the Cauchy stress tensor.

4. Constitutive equation

The constitutive state variables characterising the finite strain of an isotropic elastic material are the symmetric, second-order left bi-configuration stress tensor $\mathbf{S}$ and the symmetric, second-order left bi-configuration strain tensor $\mathbf{E}$. Central to the formulation of the constitutive stress-finite strain relation for a class of isotropic perfectly elastic solids is the condition that, for sufficiently small strains, the proposed constitutive equation must reduce to the stress-strain relationship of the classical, infinitesimal, linearized theory of elasticity, that is, the generalized Hooke's law.

4.1. Basic constitutive assumption

The basic constitutive assumption connects the stress tensor $\mathbf{S}$ to $\mathbf{E}$ by the constitutive equation

$$\mathbf{S} = \lambda(\mathrm{tr}\mathbf{E})\mathbf{I} + 2G\mathbf{E}, \tag{4.1}$$

where the material response coefficients λ and G are scalar functions of the principal invariants $\mathrm{I}_{\mathbf{E}}$, $\mathrm{II}_{\mathbf{E}}$, $\mathrm{III}_{\mathbf{E}}$ of $\mathbf{E}$.

Substituting the small strain approximations for $\mathbf{E}$ and $\mathbf{S}$ given in (2.24) and (3.17) in (4.1), we have

$$\mathbf{T} = \lambda_0(\mathrm{tr}\tilde{\mathbf{e}})\mathbf{I} + 2G_0\tilde{\mathbf{e}}, \tag{4.2}$$

where

$$\lambda_0 = \lim_{\mathbf{E}\to\tilde{\mathbf{e}}} \lambda = \text{ const.}, \qquad G_0 = \lim_{\mathbf{E}\to\tilde{\mathbf{e}}} G = \text{ const.} \tag{4.3}$$

Equation (4.2) is the stress-strain relationship of the classical, infinitesimal, linearized theory of elasticity, and hence it follows that the constitutive equation (4.1) satisfies the condition that for sufficiently small strains it must reduce to the generalized Hooke's law. Since the constitutive equation (4.1) is effectively a finite strain form of generalized Hooke's law, λ and G will be referred to as the generalised Lamé coefficients.

For incompressible materials, the stress must be replaced by the extra stress $\mathbf{S}_{\mathrm{e}} = \mathbf{S} + P\mathbf{I}$, where the isotropic stress P is kinematically indeterminate. Hence, for an incompressible isotropic elastic material, the constitutive equation (4.1) is replaced by

$$\mathbf{S} = -\bar{P}\mathbf{I} + 2G\mathbf{E} \tag{4.4}$$

it being noted that the spherical term $\lambda(\mathrm{tr}\mathbf{E})\mathbf{I}$ has been absorbed into the constraint stress $-P\mathbf{I}$.

Since $\det\boldsymbol{\Phi} \neq 0$, the symmetric kinematic tensor $\boldsymbol{\Phi}$ defined in (3.10) has an inverse and the constitutive equation (4.1) can be expressed in the form

$$\mathbf{T} = \lambda(\mathrm{tr}\mathbf{E})\boldsymbol{\Phi}^{-1} + 2G\boldsymbol{\Phi}^{-1}\mathbf{E}. \tag{4.5}$$

4.2. The Lode relation

Using a prime to denote the deviatoric part, both (4.1) and (4.4) can be expressed in the form

$$\mathbf{S}' = 2G\mathbf{E}'. \tag{4.6}$$

It follows from the first equation in (2.11) that the deviator of $\mathbf{E}$ has the spectral representation

$$\mathbf{E}' = \sum_{r=1}^{3} \varepsilon_r{}' \boldsymbol{q}_r \otimes \boldsymbol{q}_r, \tag{4.7}$$

where $\varepsilon_i'(i = 1, 2, 3)$ are the eigenvalues of $\mathbf{E}'$. In (4.7), the orthonormal triplet $\boldsymbol{q}_i$ specifies the principal axes at $\boldsymbol{x}$ common to both $\mathbf{E}$ and its deviator. Entering the spectral representation of $\mathbf{E}'$ given in (4.7) into (4.6) we obtain the relation

$$s_i{}' = 2G\varepsilon_i{}' \quad (i = 1,\ 2,\ 3), \tag{4.8}$$

and establishes that $\mathbf{S}'$ has the spectral representation

$$\mathbf{S}' = \sum_{r=1}^{3} s_r{}' \boldsymbol{q}_r \otimes \boldsymbol{q}_r, \tag{4.9}$$

where $s_i'(i = 1, 2, 3)$ are the eigenvalues of $\mathbf{S}'$; the principal axes at $\boldsymbol{x}$ common to $\mathbf{S}'$ and $\mathbf{E}'$ are defined by the orthonormal triplet $\boldsymbol{q}_i$. It follows from (4.7) and (4.9) that $\mathbf{S}'$ is coaxial with $\mathbf{E}'$.

Associated with any symmetric second-order tensor is the Lode [4] parameter facilitating the formulation of the predictions of a proposed constitutive theory. The Lode parameters for $\mathbf{S}'$ and $\mathbf{E}'$ are

$$\mu = \frac{3s_1{}'}{s_3{}' - s_2{}'}, \qquad \nu = \frac{3\varepsilon_1{}'}{\varepsilon_3{}' - \varepsilon_2{}'}. \tag{4.10}$$

With regard to (4.6) and (4.8), μ will be referred to as the Lode stress parameter, and ν as the Lode strain parameter.

The expression for $s_i{}'$ can be entered from (4.8) into the first equation of (4.10) to give

$$\mu = \nu, \tag{4.11}$$

where use has been made of the second equation in (4.10). The equality given in (4.11) will be referred to as the ‘Lode relation’.

4.3. NONLINEAR STRESS-STRAIN RESPONSE

The nonlinear stress-strain response is described by the generalised Lamé coefficient G through its dependence on the principal invariants $\mathrm{I}_{\mathbf{E}}$, $\mathrm{II}_{\mathbf{E}}$, $\mathrm{III}_{\mathbf{E}}$ of $\mathbf{E}$. It follows from (4.6) that

$$\mathrm{S}_2{}' = 4G^2\mathrm{E}_2{}', \tag{4.12}$$

where

$$\mathrm{S}_2{}' = \tfrac{1}{2}\mathrm{tr}\mathbf{S}'^2 = \tfrac{1}{12}(3+\mu^2)(s_3' - s_2')^2, \quad \mathrm{E}_2{}' = \tfrac{1}{2}\mathrm{tr}\mathbf{E}'^2 = \tfrac{1}{12}(3+\nu^2)(\varepsilon_3' - \varepsilon_2')^2 \tag{4.13}$$

are the second principal moments of $\mathbf{S}'$ and $\mathbf{E}'$, respectively.

Equation (4.12) is central to the concept of a generalised loading-response relation formulated in terms of an effective stress

$$\Phi = \sqrt{3\mathrm{S}_2{}'} = \tfrac{1}{2}(s_3{}' - s_2{}')\sqrt{3+\mu^2} = \tfrac{3}{2}\mathrm{s}_1{}'\sqrt{\frac{3+\mu^2}{\mu^2}}, \tag{4.14}$$

and effective strain

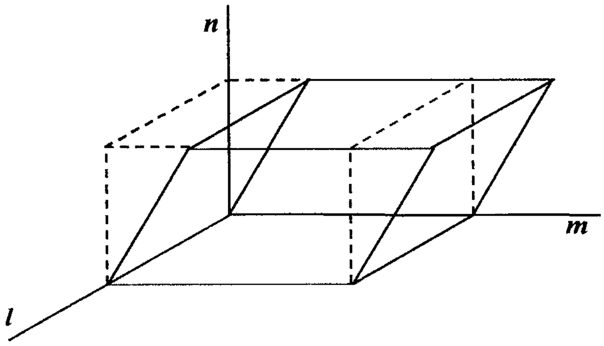

Figure 1. Initial (dashed lines) and deformed (solid lines) configurations of a material body in simple shear.

$$\epsilon = \tfrac{2}{3}\sqrt{3\mathrm{E}_2{}'} = \tfrac{1}{3}(\varepsilon_3{}' - \varepsilon_2{}')\sqrt{3+\nu^2} = \varepsilon_1{}'\sqrt{\frac{3+\nu^2}{\nu^2}}. \tag{4.15}$$

Entering the relation for s_1' from (4.8) into (4.14) and using the Lode relation we establish the generalised loading-response relation

$$\Phi = 3G\,\epsilon, \tag{4.16}$$

where use has been made of (4.15).

5. Predictions from constitutive theory

Central to the derivation of the constitutive equation (4.1) is the question as to whether it yields the general predictions identified in Section 1, as well as quantitative predictions that can be subjected to the test of experiment.

5.1. Poynting effect

Consider an initially cuboid-shaped material body $\mathscr{B}$ aligned in the reference configuration $\mathrm{B_r}$ with the Cartesian coordinate axes (X_1, X_2, X_3). Let the Cartesian coordinates (x_1, x_2, x_3) be the spatial coordinates in the deformed configuration. A simple shear deformation of amount γ is given by

$$x_1 = X_1, \quad x_2 = X_2 + \gamma X_3, \quad x_3 = X_3. \tag{5.1}$$

The effect of this homogeneous deformation is shown in Figure 1, and is seen to transform $\mathscr{B}$ to a parallelepiped having two of its faces orthogonal to the directions represented by the unit vectors $\boldsymbol{l}$ and $\boldsymbol{n}$. The direction of shear, defined by a unit vector $\boldsymbol{m}$, forms with $\boldsymbol{l}$ and $\boldsymbol{n}$ an orthonormal set. From (2.2) and (5.1)

$$\mathbf{F} = \boldsymbol{l}\otimes\boldsymbol{l} + \boldsymbol{m}\otimes\boldsymbol{m} + \boldsymbol{n}\otimes\boldsymbol{n} + \gamma\boldsymbol{m}\otimes\boldsymbol{n}. \tag{5.2}$$

From (5.2), $\det\mathbf{F} = 1$, and the deformation is isochoric. The inverse of $\mathbf{F}$ is the adjugate of $\mathbf{F}^{\mathrm{T}}$, and hence (5.2) gives

$$\mathbf{F}^{-1} = \boldsymbol{l}\otimes\boldsymbol{l} + \boldsymbol{m}\otimes\boldsymbol{m} + \boldsymbol{n}\otimes\boldsymbol{n} - \gamma\boldsymbol{m}\otimes\boldsymbol{n}. \tag{5.3}$$

Substituting in (2.10) for $\mathbf{F}$ and its inverse from (5.2) and (5.3), we obtain for the left finite strain tensor the expression

$$\mathbf{E} = \tfrac{1}{4}\gamma^2 \boldsymbol{m}\otimes\boldsymbol{m} - \tfrac{1}{4}\gamma^2 \boldsymbol{n}\otimes\boldsymbol{n} + \tfrac{1}{2}\gamma(\boldsymbol{m}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes\boldsymbol{m}). \tag{5.4}$$

Entering the form for **E** given in (5.4) into the constitutive equation (4.1) we have

$$\mathbf{S} = \tfrac{1}{2}\gamma^2 G\,\boldsymbol{m}\otimes\boldsymbol{m} - \tfrac{1}{2}\gamma^2 G\,\boldsymbol{n}\otimes\boldsymbol{n} + \gamma G\,(\boldsymbol{m}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes\boldsymbol{m}), \tag{5.5}$$

which establishes that the only non-zero components of stress are the normal components

$$S_{(\boldsymbol{m})} = \boldsymbol{m}\cdot(\boldsymbol{S}\boldsymbol{m}) = \tfrac{1}{2}\gamma^2 G, \quad S_{(\boldsymbol{n})} = \boldsymbol{n}\cdot(\boldsymbol{S}\boldsymbol{n}) = -\tfrac{1}{2}\gamma^2 G, \tag{5.6}$$

and the components of shear stress

$$S_{(\boldsymbol{m}\,\boldsymbol{n})} = \boldsymbol{m}\cdot(\boldsymbol{S}\boldsymbol{n}) = \gamma G, \quad S_{(\boldsymbol{n}\,\boldsymbol{m})} = \boldsymbol{n}\cdot(\boldsymbol{S}\boldsymbol{m}) = \gamma G. \tag{5.7}$$

From (5.6) and (5.7),

$$S_{(\boldsymbol{m})} - S_{(\boldsymbol{n})} = \gamma S_{(\boldsymbol{m}\,\boldsymbol{n})}, \tag{5.8}$$

which is independent of the generalised Lamé coefficient and consequently referred to as a universal relation.

Substitution in (3.10) for **F** from (5.2) gives for **A** the expression

$$\mathbf{A} = (1+\tfrac{1}{2}\gamma^2)\boldsymbol{l}\otimes\boldsymbol{l} + \boldsymbol{m}\otimes\boldsymbol{m} + (1+\tfrac{1}{2}\gamma^2)\boldsymbol{n}\otimes\boldsymbol{n} - \tfrac{1}{2}\gamma(\boldsymbol{m}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes\boldsymbol{m}), \tag{5.9}$$

whose derivation relies on the first equation of (2.8). Since $\boldsymbol{\Phi}$ and **T** commute, it follows from (3.9), (3.10), and (5.5) that

$$\mathbf{S}^2 = (1+\tfrac{1}{4}\gamma^2)\gamma^2 G^2 \boldsymbol{m}\otimes\boldsymbol{m} + (1+\tfrac{1}{4}\gamma^2)\gamma^2 G^2 \boldsymbol{n}\otimes\boldsymbol{n} = \mathbf{A}\mathbf{T}^2. \tag{5.10}$$

From (5.10) then

$$\mathbf{T}^2 = \mathbf{A}^{-1}\mathbf{S}^2 = (1+\tfrac{1}{2}\gamma^2)\gamma^2 G^2 \boldsymbol{m}\otimes\boldsymbol{m} + \gamma^2 G^2 \boldsymbol{n}\otimes\boldsymbol{n} + \tfrac{1}{2}\gamma^3 G^2(\boldsymbol{m}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes\boldsymbol{m}), \tag{5.11}$$

where the inverse of **A** has been obtained from (5.9).

If the same notation for the components of **T** is used, it follows from entering the form for **E** given in (5.4) into the constitutive equation (4.5) that the only non-zero components of the Cauchy stress are the normal components $T_{(\boldsymbol{m})}$, $T_{(\boldsymbol{n})}$ and the components of shear stress $T_{(\boldsymbol{m}\,\boldsymbol{n})} = T_{(\boldsymbol{n}\,\boldsymbol{m})}$. Hence, the only non-zero components of $\mathbf{T}^2$ follow from (5.11) in the form

$$(T^2)_{(\boldsymbol{m})} = (T_{(\boldsymbol{m})})^2 + (T_{(\boldsymbol{m},\boldsymbol{n})})^2 = \boldsymbol{m}\cdot(\mathbf{T}^2\boldsymbol{m}) = (1+\tfrac{1}{2}\gamma^2)\gamma^2 G, \tag{5.12}$$

$$(T^2)_{(\boldsymbol{n})} = (T_{(\boldsymbol{n})})^2 + (T_{(\boldsymbol{m}\,\boldsymbol{n})})^2 = \boldsymbol{n}\cdot(\mathbf{T}^2\boldsymbol{n}) = \gamma^2 G, \tag{5.13}$$

$$(T^2)_{(\boldsymbol{m}\,\boldsymbol{n})} = T_{(\boldsymbol{m}\,\boldsymbol{n})}(T_{(\boldsymbol{m})} + T_{(\boldsymbol{n})}) = \boldsymbol{m}\cdot(\mathbf{T}^2\boldsymbol{n}) = \tfrac{1}{2}\gamma^3 G. \tag{5.14}$$

From (5.12) - (5.14)

$$\frac{(T^2)_{(\boldsymbol{m})} - (T^2)_{(\boldsymbol{n})}}{(T^2)_{(\boldsymbol{m}\,\boldsymbol{n})}} = \frac{T_{(\boldsymbol{m})} - T_{(\boldsymbol{n})}}{T_{(\boldsymbol{m}\,\boldsymbol{n})}} = \gamma. \tag{5.15}$$

This result is of physical significance because it is independent of the generalised Lamé coefficient G, and (5.15) is therefore a universal relation. Equation (5.15) shows that the shear stress $T_{(\boldsymbol{m}\,\boldsymbol{n})}$ arises directly from the normal stress-difference $T_{(\boldsymbol{m})} - T_{(\boldsymbol{n})}$. And since it is a universal relation, it follows that the shear stress is produced in exactly the same way in every isotropic elastic solid.

Moreover, it is evident from (5.15) that the normal stresses $T_{(\boldsymbol{m})}$ and $T_{(\boldsymbol{n})}$ are unequal, a property of the material referred to as the Poynting effect.

The Cauchy stress - Cauchy-Green deformation equation also yields (5.15), in accord with the Poynting effect being regarded as a general prediction which any theoretically admissable constitutive theory must predict (see, for example [6, pp. 181–184]).

5.2. Biaxial stretching of a thin sheet

Consider a cuboid-shaped body $\mathscr{B}$ in the form of a thin sheet regarded as an incompressible, perfectly elastic solid, for which only isochoric deformations are possible;

$$J = \det\mathbf{F} = \lambda_1\lambda_2\lambda_3 = 1. \tag{5.16}$$

The material is taken to be isotropic relative to its undeformed and unstressed state, corresponding to the natural configuration.

The initially square body has parallel edges aligned with the Cartesian coordinate system, (X^1, X^2, X^3), which are the referential coordinates of $\mathscr{B}$ in the natural state. Assume a pure homogeneous deformation such that

$$x^1 = \lambda_1 X^1, \qquad x^2 = \lambda_2 X^2, \qquad x^3 = \lambda_3 X^3, \tag{5.17}$$

where the Cartesian coordinates (x^1, x^2, x^3) are the spatial coordinates in the deformed configuration. The condition of incompressibility in the form of (5.16) implies that only two of the principal stretches are independently assignable: these will be taken to be λ_2 and λ_3.

With the major surfaces free from applied stress, the only non-zero components of the left bi-configuration stress are taken to be the normal components

$$S_{22} = s_2, \qquad S_{33} = s_3 \geq 0, \qquad (S_{11} = s_1 = 0) \tag{5.18}$$

for all $\lambda_3 \geq 1$. With $s_1 = 0$ for all $s_3 \geq 0$, (this condition is used in Section 6 where predictions are compared with observed material response), expressions for the non-zero components of stress follow from (3.7) and (3.12) as

$$s_2 = \sqrt{\frac{1+\lambda_2^2\lambda_3^4}{2\lambda_2^2\lambda_3^2}}\,t_2, \qquad s_3 = \sqrt{\frac{1+\lambda_2^4\lambda_3^2}{2\lambda_2^2\lambda_3^2}}\,t_3. \tag{5.19}$$

In deriving (5.19), we have made use of (5.16).

With $s_1 = 0$ for all $s_3 \geq 0$, it follows from (4.14) and (4.15) that

$$\Phi = \sqrt{(s_3 - s_2)^2 + s_2 s_3}, \qquad \epsilon = \tfrac{1}{\sqrt{3}}\sqrt{(\varepsilon_3 - \varepsilon_2)^2 + \tfrac{1}{3}(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3)^2}. \tag{5.20}$$

Some general properties of the stress-finite strain behaviour of materials can be deduced in the context of four simple modes of deformation. Each of these modes of deformation is identified by a value of the Lode parameter. These are: $\mu(=\nu) = 0, -1, -3, -\infty$. With $s_1 = 0$ for all $s_3 \geq 0$, the expressions for μ and ν given in (4.10) can be expressed in the form

$$\mu = -\left(1 + \frac{2s_2}{s_3 - s_2}\right), \qquad \nu = -\left[1 + \frac{2(1+\lambda_3^2)(\lambda_2^4\lambda_3^2 - 1)}{(1+\lambda_2^2\lambda_3^2)(\lambda_3^2 - \lambda_2^2)}\right]. \tag{5.21}$$

In deriving (5.21), we have used (5.16).

From (5.21) it follows with use of (5.16) that

(i) if $\mu = \nu = 0$, then

$$\lambda_1 = 1, \quad \lambda_3 = \lambda_2^{-1} \geq 1, \qquad s_1 = 0, \quad s_3 = -s_2 \geq 0. \tag{5.22}$$

The conditions shown in (5.22) characterise the material response in a pure shear mode of deformation. Since a thin sheet can only be subjected to tensile edge tractions, condition $s_2 < 0$ can not be realised experimentally.

(ii) if $\mu = \nu = -1$, then

$$\lambda_1 = \lambda_2 = 1/\sqrt{\lambda_3}, \qquad s_1 = 0, \quad s_2 = 0, \qquad s_3 \geq 0. \tag{5.23}$$

The conditions shown in (5.23) characterise the material response in simple uniaxial extension, and do not provide a specific prediction regarding the material's behaviour.

(iii) if $\mu = \nu = -3$, then

$$\lambda_2 = 1, \qquad \lambda_3 = \lambda_1^{-1} \geq 1, \qquad s_1 = 0, \qquad s_3 = 2s_2 \geq 0. \tag{5.24}$$

The conditions given in the first and second equations of (5.24) for the principal stretches are formally identical to those given in the first and second equations of (5.22), (differing only in the disposition of the indices), and hence these two simple modes of deformation are formally equivalent. Equation (5.24) characterises the material's response in the pure shear mode of deformation described by Rivlin and Saunders [7] and by Jones and Treloar [8].

(iv) if $\mu = \nu = -\infty$, then

$$\lambda_2 = \lambda_3 = 1/\sqrt{\lambda_1}, \qquad s_1 = 0, \qquad s_2 = s_3 \geq 0. \tag{5.25}$$

Equation (5.25) characterises the material's response in simple equi-biaxial extension. Since any theoretically admissible stress-strain relation for isotropic, perfectly elastic materials would be expected to yield this prediction, the equi-biaxial prediction formulated in (5.25) is not significant with respect to comparing constitutive theory with the observed behaviour of a material.

The condition formulated in (5.24) gives a specific prediction which can be subjected to the test of experiment. A further prediction is that the individual stress-finite strain relations obtained from biaxial stretching, that is, the (Φ, ϵ) curves obtained from (4.14) and (4.15) and characterised by variable $\mu(=\nu)$, together with the individual (Φ, ϵ) relations associated with $\mu(=\nu) = -3, -\infty$ should compound together by way of (4.16) to give a single, generalised, stress-finite strain relation.

6. Observed material response

Of the available experimental studies, there would appear to be only three sets of measurements given in sufficient detail for subjecting the stress-strain predictions from constitutive theory to the test of experiment. These are the studies of Jones and Treloar [8], James *et al.* [9], and Rivlin and Saunders [7].

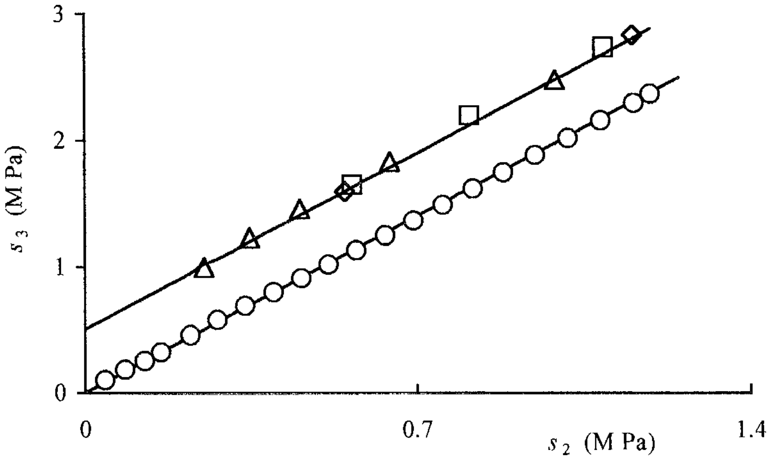

Figure 2. Variation of s_3 with s_2: ○, from Table 2 of [8], △, from Table 2 of [9], □, from Table 1 of [7], ◇ , from Table 2 of [7].

6.1. Experimental method

The experimental studies are concerned with various tests in pure homogeneous strain. The stretch $\lambda_2(\geq 1)$ was first adjusted to a fixed extension, and then the tensile stress $t_3(\geq 0)$ was incrementally increased, the tensile stress $t_2(\geq 0)$ being concomitantly adjusted to maintain the fixed stretch λ_2. In the studies of Jones and Treloar [8], the numerical values of t_2 , t_3 , λ_2 and λ_3 have been obtained from Table 2 of Haines and Wilson [10].

These three sets of measurements use a cuboid-shaped specimen in the form of a thin rubber sheet regarded as an incompressible material for which only isochoric deformations are possible. For all three studies, the material used was a sulphur-cured, natural rubber vulcanizate (smoked sheet). The three materials differ slightly in the mix employed. The material is taken to be isotropic relative to its undeformed and unstressed state, which is the natural configuration. The stress-strain relations for these experimental studies have been considered in Section 5. The present discussion is restricted to measurements made on a single specimen. The uniaxial extension measurements of Jones and Treloar [8] are not discussed because the specimen is the same as that used in the biaxial measurements. The unstressed edges may contribute an edge effect arising from the presence of the holes along these two edges, which are required for the biaxial measurements but are not in use for the uniaxial measurements.

6.2. Pure shear prediction

The conditions for pure shear characterised by $\mu = \nu = -3$ are given in Section 5(iii). Using (5.19), we show the variation of s_3 with s_2 in Figure 2. The origin of the experimental (s_3, s_2) relation for the experimental studies of James *et al.* [9], and Rivlin and Saunders [7] has been shifted so as to give an interval of 0·5 of a unit of s_3 between the two (s_3, s_2) curves. For the experimental studies of Rivlin and Saunders [7], values of t_2 , t_3 and λ_3, for $\lambda_2 = 1$ have been obtained from their Tables 1 and 2 using standard methods of extrapolation. For all three sets of measurements,

$$s_3 = 2\mathrm{A}s_2 + \mathrm{B}. \tag{6.1}$$

Table 1. Average values of Lode's bi-configuration stress parameter

	A	B	$-\mu$
[8]	0·987	0·004	$3{\cdot}019 \pm 0{\cdot}048$
[9]	0·998	0·013	$2{\cdot}917 \pm 0{\cdot}127$
[7]	1·027	−0·005	$2{\cdot}928 \pm 0{\cdot}060$

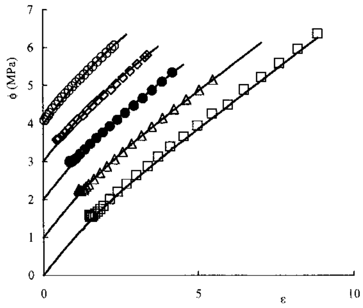

Figure 3. Variation of the effective stress Φ with the effective strain ϵ using the results given in Table 2 of [8], λ_2: ○, 1·0 ; ◇ , $1{\cdot}50_2$; ●, $1{\cdot}98_4$; △, $2{\cdot}29_5$; □, $2{\cdot}62_3$

The prediction $s_3 = 2s_2$ given in (5.24) requires A = 1, B = 0 which compare well with the experimentally determined values of A and B given in Table 1.

Entering the measured values of s_2 and s_3 into the first equation in (4.10) gives the average values of μ shown in Table 1. These values of μ are to be compared with the theoretical prediction, $\mu = -3$. The last two pairs of values of S_3 corresponding to $\lambda_2 = 3{\cdot}0$, $3{\cdot}5$ of the experimental studies of James *et al.* [9] have been omitted since for these values of λ_2 the material is no longer fully isotropic, (see Section 7).

It is evident from Figure 2 and Table 1 that all three experimental (s_3, s_2) relations are in good agreement with the theoretical prediction represented by the full straight lines.

6.3. Generalised stress-strain relation

Values of the effective stress Φ and the effective strain ϵ for the three sets of measurements have been evaluated from (5.20). The experimental (Φ, ϵ) curves are shown in Figure 3 for the results given by Jones and Treloar [8]. The origin of each (Φ, ϵ) curve corresponding to $\lambda_2 = 2{\cdot}29_5$, $1{\cdot}98_4$, $1{\cdot}50_2$, 1, has been shifted so as to give equal intervals of one unit of Φ between the individual (Φ, ϵ), curves. The five full line curves shown in Figure 3 are identical.

Shown in Figure 4 is the variation of ϕ with ϵ for the experimental studies of James *et al.* [9]. The origin of each (Φ, ϵ) curve corresponding to $\lambda_2 = 1{\cdot}3$, 1·5, 1·7, 2·0, has been shifted so as to give equal intervals of one unit of ϕ between the individual (Φ, ϵ), curves. The five full line curves shown in Figure 4 are identical.

Using the experimental results given in Table 1 of [7], the four (Φ, ϵ) curves for $I_{\mathbf{B}} = 5, 7, 9, 11$, are shown in Figure 5. For this experimental study, the results given by Rivlin and Saunders in their Table 1 are such that the (λ_2, t_2) curve increases to a maximum value, (λ_2^*, t_2^*), and the (λ_3, t_3) curve decreases to a minimum observed value, (λ_3^*, t_3^*). These two curves approach each other in such a way that each curve appears to be the continuation of the

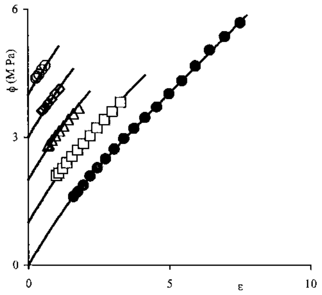

Figure 4. Variation of the effective stress Φ with the effective strain ϵ, using the results given in Table 2 of [9]: λ_2; ◦, 1·3; ◇, 1·5; △, 1·7; □, 2·0 ; •, 2·5.

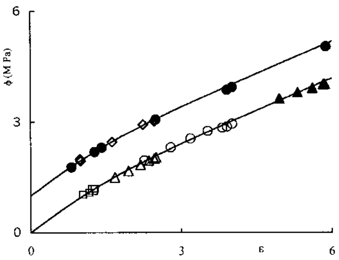

Figure 5. Variation of the effective stress ϕ with the effective strain ϵ using the results given in Table 1 of [7], $\mathbf{I}_{\boldsymbol{B}}$: □, 5; △, 7; ◦, 9; ▲, 11; ●, $\lambda_2 = \lambda_3$; ◇, $\lambda_2 = 1$.

other, so that there is no well-defined intersection of the curves at $(\lambda_2 = \lambda_3, t_2 = t_3)$. However, the differences $(\lambda_3^* - \lambda_2^*)$, $(t_3^* - t_2^*)$ are sufficiently small for the equibiaxial extension λ_{E} and the equibiaxial Cauchy stress t_{E} to be approximated with

$$\lambda_{\mathrm{E}} = \tfrac{1}{2}(\lambda_2^* + \lambda_3^*), \qquad t_{\mathrm{E}} = \tfrac{1}{2}(t_2^* + t_3^*). \tag{6.2}$$

In the case of the results given in Table 2 of [7], the two relevant sets of (λ_2, t_2), (λ_3, t_3) curves do have a well-defined intersection, giving three further values of $\lambda_{\mathrm{E}}, t_{\mathrm{E}}$. The (Φ, ϵ) curve for these equibiaxial results is shown in Figure 5 using blocked circles; the origin for this curve has been shifted by one unit of Φ. Also shown in Figure 5 are the results for the pure shear measurements obtained by extrapolation of the (λ_2, t_2), (λ_3, t_3) curves in the way described above. The (Φ, ϵ) curve for pure shear has been shifted by one unit of Φ. The two full line curves shown in Figure 5 are identical.

It is evident from Figures 3, 4 and 5 that, within the limits of experimental accuracy, the individual (Φ, ϵ) curves for each of the three sets of measurements compound together to give three generalised experimental, bi-configuration stress-finite strain curves.

All available results for the equi-biaxial mode of deformation are in good accord with the generalised experimental, bi-configuration stress-finite strain curves.

6.4. Generalised Lamé coefficient G

The generalised stress-strain relation of (4.16) involves a single material coefficient, that is, the generalised Lamé coefficient G. Taking the generalised Lamé coefficient G to be of the form

$$G = G_{\mathrm{a}} + (G_0 - G_{\mathrm{a}})\frac{\tanh(\eta\epsilon)}{\eta\epsilon}, \tag{6.3}$$

Table 2. Values of the parameters appearing in the mechanical equation of state

	ϕ_a (M Pa)	G_a (M Pa)	η	G_0 (M Pa)
[8]	0·730	0·208	0·548	0·342
[9]	0·723	0·221	0·605	0·367
[7]	0·700	0·194	0·596	0·333

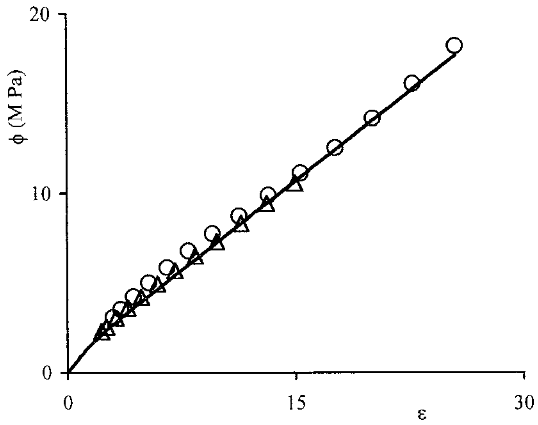

Figure 6. Variation of the effective stress Φ with the effective strain ϵ, using the results given in Table 2 of [9]: λ_2; $\triangle$, 3·0 ; $\circ$, 3·5.

and entering this form for G into (4.16) leads to the relation

$$\Phi = \phi_a \tanh(\eta\epsilon) + 3G_a\epsilon = 3G\epsilon, \tag{6.4}$$

where

$$G_0 = \lim_{\epsilon\to 0} G, \qquad \phi_a = 3\frac{(G_0 - G_a)}{\eta} \tag{6.5}$$

where G_0 , G_a , ϕ_a , η are constants characteristic of material properties.

Using the values of G_a , ϕ_a , η given in Table 2, the five identical full line curves shown in Figure 3 have been calculated from (6.4). Similarly, using the values of G_a , ϕ_a , η given in Table 2, the five full line curves shown in Figure 4, and the two full line curves shown in Figure 5, have been calculated from (6.4).

The discussion given in Sections 4, 5 and 6 has centred on the use of the proposed constitutive stress-finite strain relation formulated in the spatial description. However, since **E** and **J** have in common the principal strains ε_i, and **S** and $\boldsymbol{\Sigma}$ have in common the principal stresses s_i, it follows from (4.14) and (4.15) that both the effective stress Φ and the effective strain ϵ are independent of whether the constitutive equation is formulated in the spatial description or in the referential description. This leads by way of (4.16) to the conclusion that the generalised Lamé coefficient G is a specific property characteristic of a material.

7. Discussion

The observation that the individual stress-strain relations obtained from biaxial tests compound together to give a single generalised bi-configuration stress-finite strain curve over a wide range of finite strain is taken as evidence in support of the assumption that the materials can be regarded as isotropic over a wide range of finite strain. The individual (Φ , ϵ) curves are shown in Figure 6 for the biaxial measurements given in [9] for $\lambda_2 = 3{\cdot}0, 3{\cdot}5$. In Figure 6, the full line (Φ, ϵ) curve has been calculated from (6.4) using the values of G_a, ϕ_a, η given in Table 2.

It is seen from Figure 6 that there is a range of finite strain over which the individual (Φ, ϵ) curves fail to correlate with the full line curve calculated from (6.4). With further increase in effective strain, the individual (Φ, ϵ) curves again become indistinguishable from the full-line curve. The results shown in Figure 6 identify a particular value of the effective strain at which the (Φ, ϵ) curves first deviate from the full line (Φ, ϵ) curve. The deviation of the experimental curve from the full line curve is tentatively attributed to the material becoming progressively anisotropic with increasing strain. As the effective strain is increased towards the equibiaxial condition, $\lambda_2 = \lambda_3$, the individual (Φ,ϵ) curves shown in Figure 6 again correlate with the original generalised stress-strain curve identified as the full line curve, which is taken to imply that the material progressively regains the initial state of isotropy.

The proposed constitutive equation predicts the Poynting effect, in accord with the condition that any theoretically admissible constitutive equation must predict this effect. Moreover, the proposed constitutive equation predicts stress equality for the equi-biaxial extension mode of deformation, the prediction being accurately confirmed by the available results of experiment. For sufficiently small strains, the constitutive equation reduces to the stress-strain relationship of the classical infinitesimal linearized theory of elasticity, that is, the generalised Hooke's law. The specific prediction associated with the pure shear mode of deformation is shown to be accurately confirmed by the available results of experiment. The high degree of accuracy of the correlation between the specific (quantitative) prediction and the results of experiment for the pure shear mode of deformation confirms the technical utility of the proposed new measure of stress, that is, the bi-configuration stress tensor. The specific prediction that the individual stress-finite strain relations obtained from bi-axial stretching should compound together to give a single, generalised, characteristic stress-finite strain relation is shown to be accurately confirmed by the available results of experiment. The single, generalised, characteristic stress-finite strain relation establishes the generalised Lamé coefficients as specific, fundamental, quantitative properties of a material.

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